Belief network inference

Main approaches to determine posterior distributions in graphical models:

- Variable Elimination, recursive conditioning: exploit the structure of the network to eliminate (sum out) the non-observed, non-query variables one at a time.

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- Stochastic simulation: random cases are generated according to the probability distributions.
- Variational methods: find the closest tractable distribution to the (posterior) distribution we are interested in.
- Bounding approaches: bound the conditional probabilites above and below and iteratively reduce the bounds.
- ...

Factors

- A factor is a representation of a function from a tuple of random variables into a number.
- We write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.

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- A factor is a representation of a function from a tuple of random variables into a number.
- We write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.
- We can assign some or all of the variables of a factor:
  - $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in dom(X_1)$, is a factor on $X_2, \ldots, X_j$.
  - $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.

  The former is also written as $f(X_1, X_2, \ldots, X_j)_{X_1 = v_1}$, etc.

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

Example factors

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Y	Z	val
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t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Example factors

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t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	
f	

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
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f	f	t	0.3
f	f	f	0.7

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t	t	0.1
t	f	0.9
f	t	0.2
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$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

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f	t	0.2
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Y	val
t	0.9
f	0.8

Example factors

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X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X=t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X=t, Y, Z=f)$:

Y	val
t	0.9
f	0.8

$r(X=t, Y=f, Z=f) =$

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X=t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X=t, Y, Z=f)$:

Y	val
t	0.9
f	0.8

$$r(X=t, Y=f, Z=f) = 0.8$$

Multiplying factors

The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 * f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 * f_2)(\overline{X}, \overline{Y}, \overline{Z}) = f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	
t	f	t	
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

Summing out variables

We can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$
\begin{aligned}
&(\sum_{X_1} f)(X_2, \ldots, X_j) \\
&= \quad f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)
\end{aligned}
$$

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	
f	t	
f	f	

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	
f	f	

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Exercise

Given factors:

s:

A	val
t	0.75
f	0.25

t:

A	B	val
t	t	0.6
t	f	0.4
f	t	0.2
f	f	0.8

o:

A	val
t	0.3
f	0.1

What are the following factors over?

(a) $s * t$

(b) $\sum_A s * t$

(c) $\sum_B s * t$

(d) $\sum_A \sum_B s * t$

(e) $s * t * o$

(f) $\sum_B s * t * o$

Evidence

- If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j$:

$$P(Z \mid Y_1 = v_1, \ldots, Y_j = v_j)$$

$$=$$

Evidence

- If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j$:

$$
\begin{aligned}
& P(Z \mid Y_1 = v_1, \ldots, Y_j = v_j) \\
= \quad & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)} \\
= \quad &
\end{aligned}
$$

Evidence

- If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j$:

$$
\begin{aligned}
& P(Z \mid Y_1 = v_1, \ldots, Y_j = v_j) \\
= \; & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)} \\
= \; & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{\sum_Z P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}.
\end{aligned}
$$

- So the computation reduces to the probability of $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$.
- Normalize at the end, by summing out Z and dividing.

Probability of a conjunction

Suppose the variables of the belief network are $X_1, \ldots, X_n$.
To compute $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$, we sum out the other variables, $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} - \{Z\} - \{Y_1, \ldots, Y_j\}$.
We order the Z_i into an elimination ordering.

$$P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$$

$$=$$

Probability of a conjunction

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We order the Z_i into an elimination ordering.

$$
\begin{aligned}
& P(Z, Y_1 = v_1, \ldots, Y_j = v_j) \\
&= \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
&=
\end{aligned}
$$

Probability of a conjunction

Suppose the variables of the belief network are $X_1, \ldots, X_n$.
To compute $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$, we sum out the other variables, $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} - \{Z\} - \{Y_1, \ldots, Y_j\}$.
We order the Z_i into an elimination ordering.

$$
\begin{aligned}
& P(Z, Y_1 = v_1, \ldots, Y_j = v_j) \\
= & \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
= & \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i \mid parents(X_i))_{Y_1 = v_1, \ldots, Y_j = v_j}.
\end{aligned}
$$

Computing sums of products

Computation in belief networks reduces to computing the sums of products.

- How can we compute $ab + ac$ efficiently?

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Computing sums of products

Computation in belief networks reduces to computing the sums of products.

- How can we compute $ab + ac$ efficiently?
- Distribute out the a giving $a(b + c)$
- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i \mid parents(X_i))$ efficiently?
- Distribute out those factors that don't involve Z_1.

Variable elimination algorithm

To compute $P(Z \mid Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j)$:

- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- Sum out each of the other variables (the $\{Z_1, \ldots, Z_k\}$) according to some elimination ordering.
- Multiply the remaining factors. Normalize by dividing the resulting factor $f(Z)$ by $\sum_Z f(Z)$.

Summing out a variable

To sum out a variable Z_j from a product $f_1, \ldots, f_k$ of factors:

- Partition the factors into
  - those that don't contain Z_j, say $f_1, \ldots, f_i$,
  - those that contain Z_j, say $f_{i+1}, \ldots, f_k$

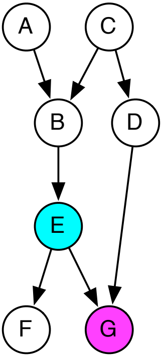
We know:

$$\sum_{Z_j} f_1 * \cdots * f_k = f_1 * \cdots * f_i * \left(\sum_{Z_j} f_{i+1} * \cdots * f_k \right).$$

- Explicitly construct a representation of the rightmost factor. Replace the factors $f_{i+1}, \ldots, f_k$ by the new factor.

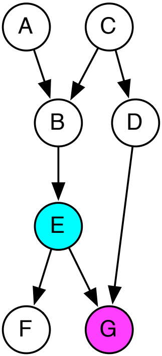
Example

$$P(E \mid g) \quad =$$

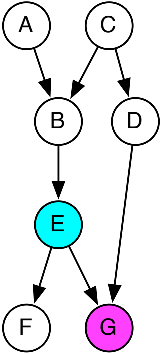


Example

$$P(E \mid g) \quad = \quad \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$



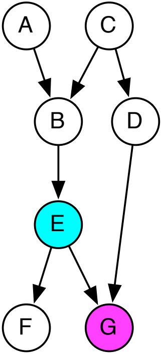
Example



$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$P(E \wedge g) = \sum_F \sum_B \sum_C \sum_A \sum_D$$

Example



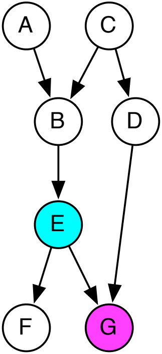
$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$P(E \wedge g)$$

$$= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC)$$
$$P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED)$$

$$=$$

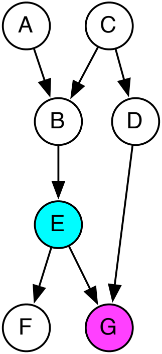
Example



$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$\begin{aligned} P(E \wedge g) & \\ = & \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC) \\ & P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED) \\ = & \\ & \left(\sum_D P(D \mid C)P(g \mid ED)\right)\Bigg) \end{aligned}$$

Example



$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

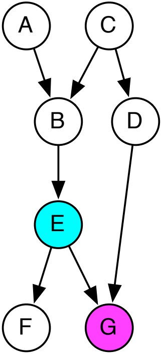
$$P(E \wedge g)$$

$$= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC)$$
$$P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED)$$

$$=$$

$$\left(\sum_A P(A)P(B \mid AC) \right)$$
$$\left(\sum_D P(D \mid C)P(g \mid ED) \right) \Bigg) \Bigg)$$

Example



$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

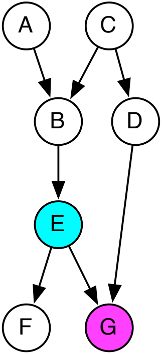
$$P(E \wedge g)$$

$$= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC)$$
$$P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED)$$

$$=$$

$$\sum_C \left(P(C) \left(\sum_A P(A)P(B \mid AC) \right) \left(\sum_D P(D \mid C)P(g \mid ED) \right) \right)$$

Example



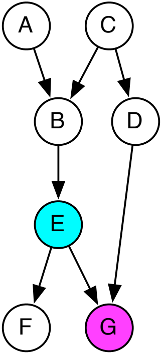
$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$P(E \wedge g)$$

$$= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC) P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED)$$

$$= \sum_B P(E \mid B) \sum_C \left(P(C) \left(\sum_A P(A)P(B \mid AC) \right) \left(\sum_D P(D \mid C)P(g \mid ED) \right) \right)$$

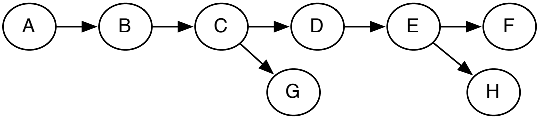
Example



$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$\begin{aligned}
P(E \wedge g) & \\
&= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC) \\
&\qquad P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED) \\
&= \left(\sum_F P(F \mid E)\right) \\
&\qquad \sum_B P(E \mid B) \sum_C \left(P(C)\left(\sum_A P(A)P(B \mid AC)\right)\right. \\
&\qquad\qquad \left.\left(\sum_D P(D \mid C)P(g \mid ED)\right)\right)
\end{aligned}$$

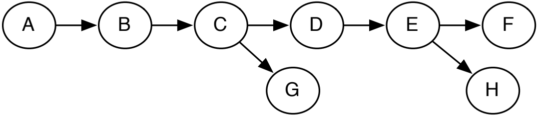
Variable Elimination example



Query: $P(G \mid f)$; elimination ordering: A, H, E, D, B, C

$$P(G \mid f) \propto$$

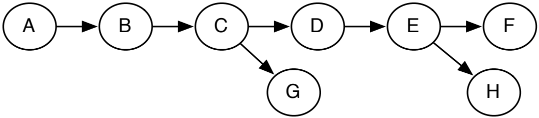
Variable Elimination example



Query: $P(G \mid f)$; elimination ordering: A, H, E, D, B, C

$$P(G \mid f) \propto \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B \mid A)P(C \mid B)$$
$$P(D \mid C)P(E \mid D)P(f \mid E)P(G \mid C)P(H \mid E)$$

Variable Elimination example



Query: $P(G \mid f)$; elimination ordering: A, H, E, D, B, C

$$P(G \mid f) \propto \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B \mid A)P(C \mid B) \\ P(D \mid C)P(E \mid D)P(f \mid E)P(G \mid C)P(H \mid E)$$

$$= \sum_C \left(\sum_B \left(\sum_A P(A)P(B \mid A) \right) P(C \mid B) \right) P(G \mid C) \\ \left(\sum_D P(D \mid C) \left(\sum_E P(E \mid D)P(f \mid E) \sum_H P(H \mid E) \right) \right)$$

Pruning Irrelevant Variables (Belief networks)

Suppose you want to compute $P(X \mid e_1 \dots e_k)$:

- Prune any variables that have no observed or queried descendents.
- Connect the parents of any observed variable.
- Remove arc directions.
- Remove observed variables.
- Remove any variables not connected to X in the resulting (undirected) graph.