# Agents as Processes

Agents carry out actions:

- forever infinite horizon
- until some stopping criteria is met indefinite horizon
- finite and fixed number of steps finite horizon

# Decision-theoretic Planning

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- actions can be stochastic; the outcome of an action can't be fully predicted
- there is a model that specifies the (probabilistic) outcome of actions and the rewards
- the world is fully observable

# Initial Assumptions

- flat or modular or hierarchical
- explicit states or features or individuals and relations
- static or finite stage or indefinite stage or infinite stage
- fully observable or partially observable
- deterministic or stochastic dynamics
- goals or complex preferences
- single agent or multiple agents
- knowledge is given or knowledge is learned
- perfect rationality or bounded rationality

# Utility and time

- Would you prefer \$1000 today or \$1000 next year?
- What price would you pay now to have an eternity of happiness?
- How can you trade off pleasures today with pleasures in the future?

# Utility and time

- How would you compare the following sequences of rewards (per week):

  A: \$1000000, \$0, \$0, \$0, \$0, \$0,...
  B: \$1000, \$1000, \$1000, \$1000, \$1000,...
  C: \$1000, \$0, \$0, \$0, \$0,...
  D: \$1, \$1, \$1, \$1, \$1,...
  E: \$1, \$2, \$3, \$4, \$5,...

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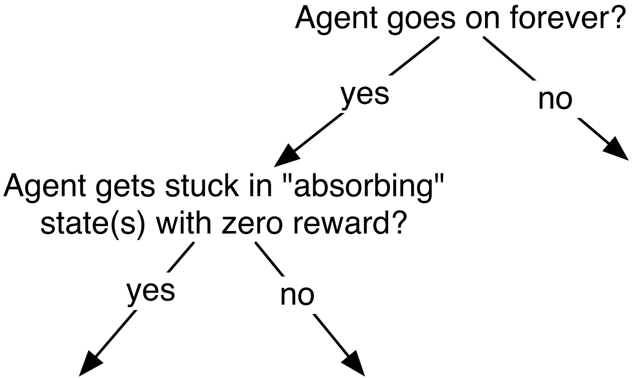
- total reward $V = \sum_{i=1}^{\infty} r_i$
- average reward $V = \lim_{n \to \infty} (r_1 + \cdots + r_n)/n$

# Average vs Accumulated Rewards



# Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \ldots$ in time.

- discounted return $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$
  $\gamma$ is the discount factor $0 \leq \gamma \leq 1$.

# Properties of the Discounted Rewards

- The discounted return for rewards $r_1, r_2, r_3, r_4, \ldots$ is

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$$V_t = r_t + \gamma V_{t+1}$$
- How is the infinite future valued compared to immediate rewards?
$1 + \gamma + \gamma^2 + \gamma^3 + \cdots = 1/(1 - \gamma)$
Therefore $\dfrac{\text{minimum reward}}{1-\gamma} \leq V_t \leq \dfrac{\text{maximum reward}}{1-\gamma}$
- We can approximate $V$ with the first $k$ terms, with error:
$$V - (r_1 + \gamma r_2 + \cdots + \gamma^{k-1} r_k) = \gamma^k V_{k+1}$$

# World State

- The world state is the information such that if the agent knew the world state, no information about the past is relevant to the future. Markovian assumption.
- $S_i$ is state at time $i$, and $A_i$ is the action at time $i$:

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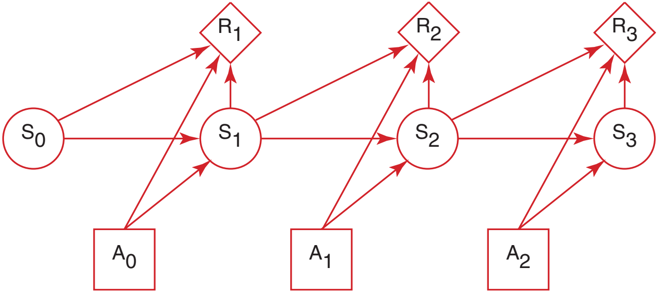
  $P(s' \mid s, a)$ is the probability that the agent will be in state $s'$ immediately after doing action $a$ in state $s$.
- The dynamics is stationary if the distribution is the same for each time point.

# Decision Processes

- A Markov decision process augments a Markov chain with actions and values:



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- $\gamma$ is discount factor.

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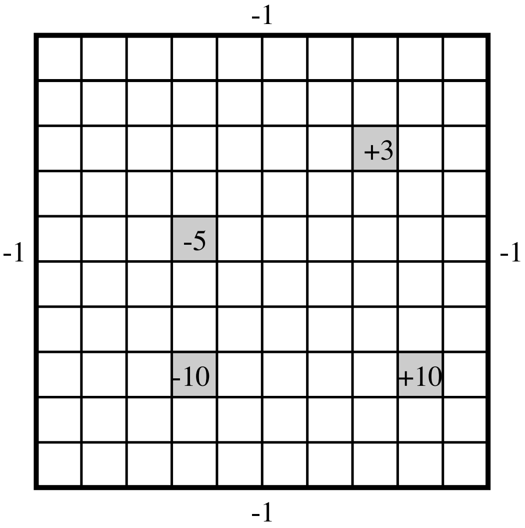
- Reward (does not depend on resulting state):

| State | Action | Reward |
|---|---|---|
| fit | exercise | 8 |
| fit | relax | 10 |
| unfit | exercise | 0 |
| unfit | relax | 5 |

# Example: Simple Grid World



# Grid World Model

- Actions: up, down, left, right.
- 100 states corresponding to the positions of the robot.
- Robot goes in the commanded direction with probability 0.7, and one of the other directions with probability 0.1.
- If it crashes into an outside wall, it remains in its current position and has a reward of $-1$.
- Four special rewarding states; the agent gets the reward when leaving.

# Planning Horizons

The planning horizon is how far ahead the planner looks to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive ($+10$ or $+3$) reward state.
  - the process never halts
  - infinite horizon

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- The robot gets flung to one of the corners at random after leaving a positive (+10 or +3) reward state.
  - the process never halts
  - infinite horizon
- The robot gets +10 or +3 in the state, then it stays there getting no reward. These are absorbing states.
  - The robot will eventually reach an absorbing state.
  - indefinite horizon

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What information is available when the agent decides what to do?

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[This lecture only considers FOMDPs]

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  Given a state $s$, $\pi(s)$ specifies what action the agent who is following $\pi$ will do.
- An optimal policy is one with maximum expected discounted reward.
- For a fully-observable MDP with stationary dynamics and rewards with infinite or indefinite horizon, there is always an optimal stationary policy.

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For the grid world with 100 states and 4 actions,
how many stationary policies are there?

# Value of a Policy

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- $Q^{\pi}$ and $V^{\pi}$ can be defined mutually recursively:

$$
\begin{aligned}
Q^{\pi}(s, a) &= \\
V^{\pi}(s) &=
\end{aligned}
$$

# $Q$, $V$, $\pi$, $R$

$$Q^\pi(s,a) = \sum_{s'} P(s' \mid a, s)\left(R(s,a,s') + \gamma V^\pi(s')\right)$$

$$= R(s,a) + \gamma \sum_{s'} P(s' \mid a, s) V^\pi(s')$$

$$V^\pi(s) = Q^\pi(s, \pi(a))$$

where

$$R(s,a) = \sum_{s'} P(s' \mid a, s) R(s,a,s')$$

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V^*(s) &= \max_a Q^*(s,a) \\
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R(s,a) = \sum_{s'} P(s' \mid a,s) R(s,a,s')
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  The error reduces proportionally to $\dfrac{\gamma^k}{1-\gamma}$

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- This converges to the optimal value functions, if each state and action is visited infinitely often in the limit.
- It can either store $V[s]$ or $Q[s, a]$.

# Asynchronous VI: storing $V[s]$

- Repeat forever:
  - Select state $s$
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# Policy Iteration

- Set $\pi_0$ arbitrarily, let $i = 0$
- Repeat:
  - evaluate $Q^{\pi_i}(s, a)$
  - let $\pi_{i+1}(s) = argmax_a Q^{\pi_i}(s, a)$
  - set $i = i + 1$
- until $\pi_i(s) = \pi_{i-1}(s)$

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Evaluating $Q^{\pi_i}(s, a)$ means finding a solution to a set of $|S| \times |A|$ linear equations with $|S| \times |A|$ unknowns.

It can also be approximated iteratively.

# Modified Policy Iteration

Set $\pi[s]$ arbitrarily
Set $Q[s,a]$ arbitrarily
Repeat forever:

- Repeat for a while:
  - Select state $s$, action $a$
  - $Q[s,a] \leftarrow \sum_{s'} P(s' \mid s,a) \left(R(s,a,s') + \gamma Q[s', \pi[s']]\right)$
- $\pi[s] \leftarrow argmax_a Q[s,a]$

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