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- Function to be minimized: the number of conflicts.

Iterative Best Improvement (2 stage) “greedy descent”

- Start with random assignment (for each variable, select a value for that variable at random)
- Repeat:
  - Select a variable that participates in the most conflicts
  - Select a different value for that variable
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All selections are random and uniform.

Any Conflict

- Start with random assignment (for each variable, select a value for that variable at random)
- Repeat:
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- How can you compare three algorithms when
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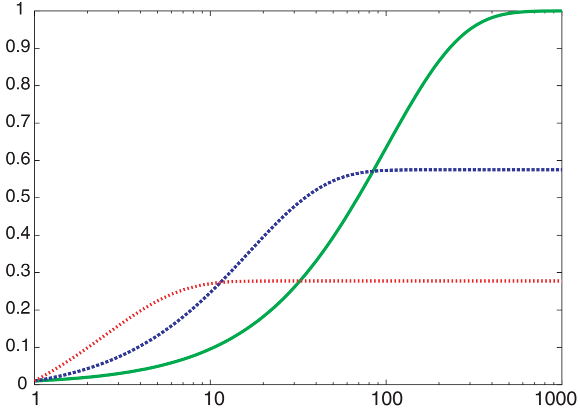
- Summary statistics, such as mean run time, median run time, and mode run time don't make much sense.

Runtime Distribution

x-axis runtime (or number of steps)

y-axis the proportion (or number) of runs that are solved within that runtime



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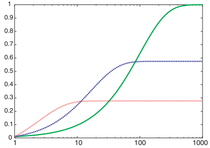

- Do this this a few times to gauge the variability (take a statistics course!)
- Sometimes use number of steps instead of run time (because computers measure small run times inaccurately) . . . not good measure to compare algorithms if steps take different times

Randomized Algorithms

- A probabilistic mix of *greedy* and *any-conflict* — e.g., 70% of time pick best variable, otherwise pick any variable in a conflict – works better than either alone.

Stochastic Local Search

Stochastic local search is a mix of:

- Greedy descent: pick the best variable and/or value
- Random walk: picking variables and values at random
- Random restart: reassigning values to all variables

Some of these might be more complex than the others.
A probabilistic mix might work better.

Greedy Descent Variants

To select a variable to change and a new value for it:

- Most Improving Step: Find a variable-value pair that minimizes the number of conflicts.

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- Why would we?
  Because some are easier to solve than other. E.g., in scheduling exams....
- If A is a total assignment, define $h(A)$ to be a measure of the difficulty of solving problem from A.
- $h(A) = 0$ then A a solution; lower h is better

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Temperature	1-worse	2-worse	3-worse
10	0.91	0.81	0.74
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- Temperature can be reduced.

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n	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.8$
5	0.410	0.832	0.969	0.9997
10	0.65	0.971	0.9990	0.9999998
20	0.878	0.9992	0.9999991	0.9999999999
50	0.995	0.99999998	0.9999999999999991	1.0

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- It can be expensive if k is large.

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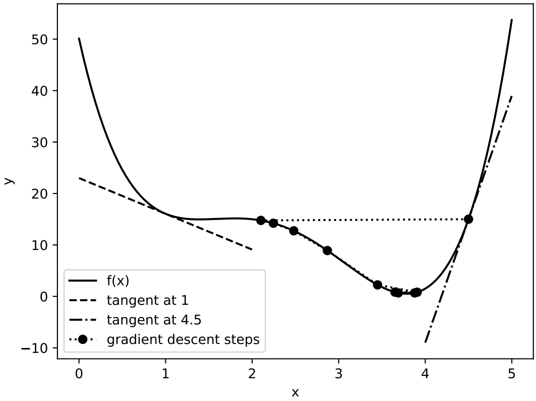
  η is the step size.
- Neural networks do gradient descent with many parameters (variables) to minimize an error on a dataset. Some large language models have over 10^{12} parameters.

Gradient Descent



$$y = 2 * (x - 1.3) * (x - 1.5) * (x - 2) * (x - 4.5) + 15$$

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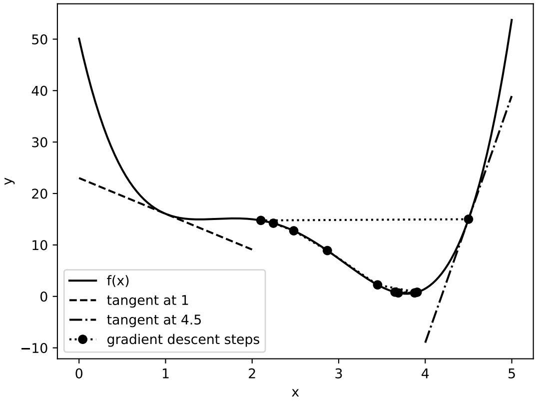

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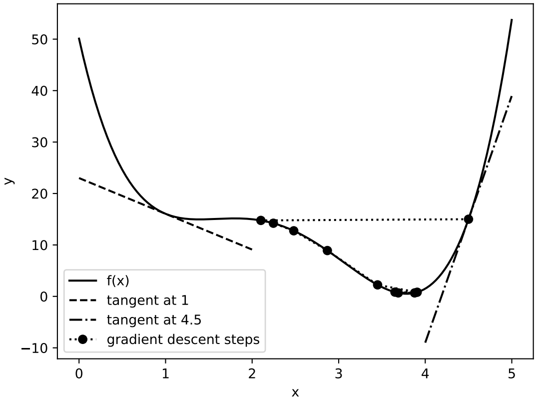

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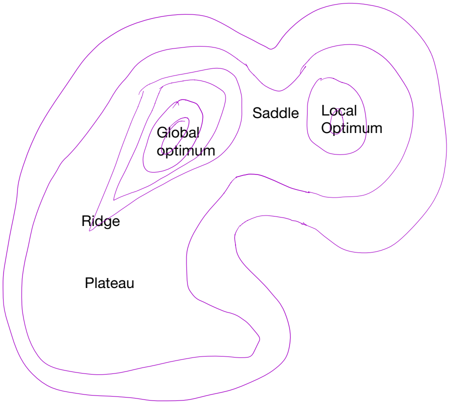

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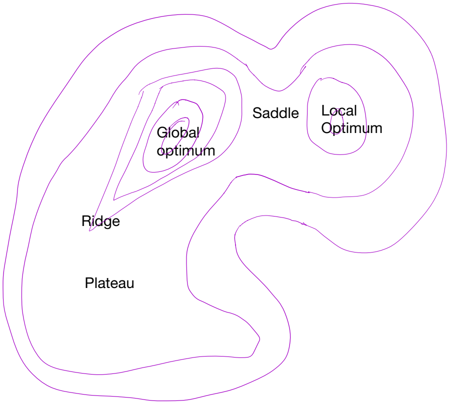

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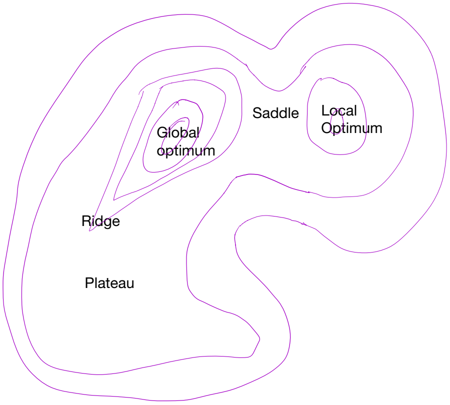

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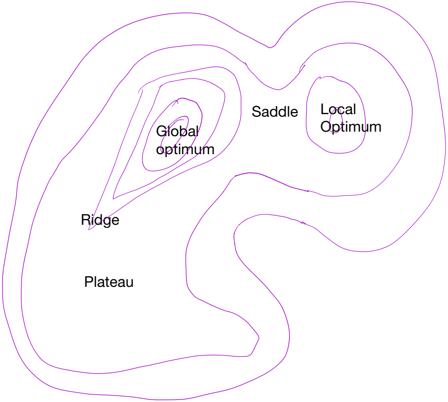
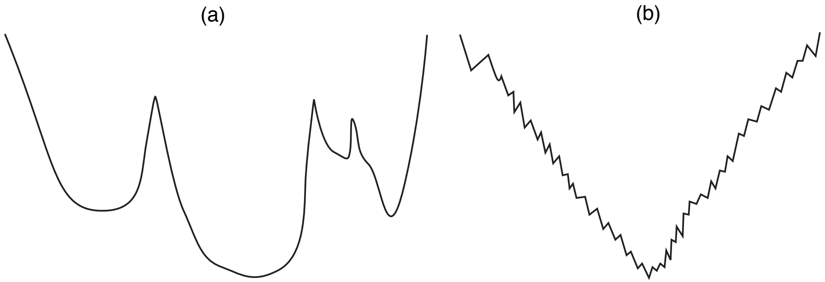


1-Dimensional Ordered Examples

Two 1-dimensional search spaces; small step right or left:



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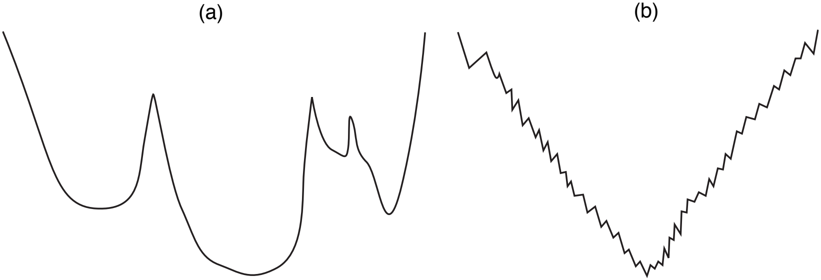

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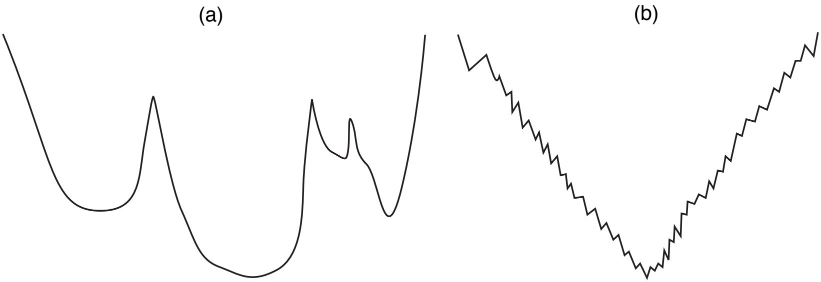

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- Like k restarts, but uses k times the *minimum* number of steps.

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- Like asexual reproduction: each individual mutates and the fittest ones survive.

Genetic Algorithms

- Like stochastic beam search, but pairs of individuals are combined to create the offspring.
- For each generation:
  - Randomly choose pairs of individuals where the fittest individuals are more likely to be chosen.
  - For each pair, perform a crossover: form two offspring each taking different parts of their parents.

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  - Mutate some values.
- Stop when a solution is found.

Crossover

- Given two individuals:

$$X_1 = a_1, X_2 = a_2, \ldots, X_m = a_m$$

$$X_1 = b_1, X_2 = b_2, \ldots, X_m = b_m$$

- Select i at random.
- Form two offspring:

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- The effectiveness depends on the ordering of the variables.
- Many variations are possible.

Optimization

An optimization problem is given

- a set of variables, each with an associated domain
- an objective function that maps total assignments to real numbers, and
- an optimality criterion, which is typically to find a total assignment that minimizes (or maximizes) the objective function.

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- Problem: we can't tell if a value is a global minimum unless we do systematic search