Local Search

Local Search:

- Maintain a complete assignment of a value to each variable.
- Start with random assignment or a best guess.
- Repeat:
  - Select a variable to change
  - Select a new value for that variable
- Until a satisfying assignment is found

Local Search for CSPs

- Aim: find an assignment with zero unsatisfied constraints.
- Given an assignment of a value to each variable, a conflict is an unsatisfied constraint.
- The goal is an assignment with zero conflicts.
- Function to be minimized: the number of conflicts.

Iterative Best Improvement (2 stage) “greedy descent”

- Start with random assignment (for each variable, select a value for that variable at random)
- Repeat:
  - Select a variable that participates in the most conflicts
  - Select a different value for that variable
- Until a satisfying assignment is found

All selections are random and uniform.

Any Conflict

- Start with random assignment (for each variable, select a value for that variable at random)
- Repeat:
  - Select a variable at random that participates in any conflict
  - Select a different value for that variable
- Until a satisfying assignment is found

All selections are random and uniform.

Comparing Stochastic Algorithms

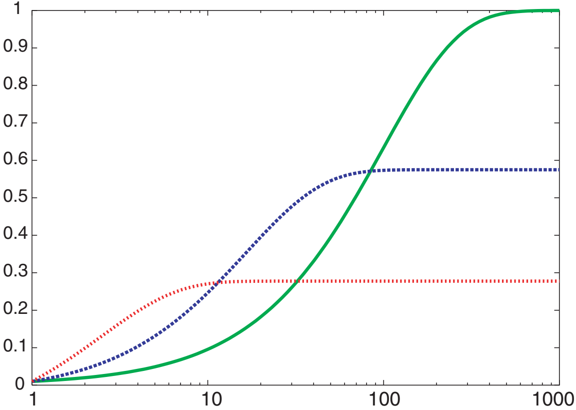
Which of the preceding algorithms work better?
How would we tell if one is better than the other?

- How can you compare three algorithms when
  - one solves the problem 30% of the time very quickly but doesn't halt for the other 70% of the cases
  - one solves 60% of the cases reasonably quickly but doesn't solve the rest
  - one solves the problem in 100% of the cases, but slowly?
- Summary statistics, such as mean run time, median run time, and mode run time don't make much sense.

Runtime Distribution

x-axis runtime (or number of steps)

y-axis the proportion (or number) of runs that are solved within that runtime

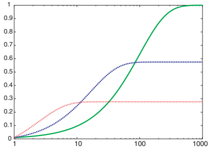


Runtime Distribution

- Run the same algorithm on the same instance for a number of trials (e.g., 100 or 1000)
- Sort the trials according to the run time.
- Plot:

  x-axis run time of the trial
  y-axis index of the trial

  This produces a cumulative distribution



- Do this this a few times to gauge the variability (take a statistics course!)
- Sometimes use number of steps instead of run time (because computers measure small run times inaccurately) . . . not good measure to compare algorithms if steps take different times

Randomized Algorithms

- A probabilistic mix of *greedy* and *any-conflict* — e.g., 70% of time pick best variable, otherwise pick any variable in a conflict – works better than either alone.

Stochastic Local Search

Stochastic local search is a mix of:

- Greedy descent: pick the best variable and/or value
- Random walk: picking variables and values at random
- Random restart: reassigning values to all variables

Some of these might be more complex than the others.
A probabilistic mix might work better.

Greedy Descent Variants

To select a variable to change and a new value for it:

- Most Improving Step: Find a variable-value pair that minimizes the number of conflicts.
  What data structures are required?
- Two Stage Choice: Select a variable that participates in the most conflicts.
  Select a value that minimizes the number of conflicts.
  What data structures are required?
- Any Conflict: Select a variable that appears in any conflict.
  Select a value at random.
  What data structures are required?
- Select a variable at random.
  Select a value that minimizes the number of conflicts.
  What data structures are required?
- Select a variable and value at random; accept this change if it doesn't increase the number of conflicts.
  What data structures are required?

Steps

- One measure of an assignment is number of conflicts
- It is possible to weight some conflicts higher than others.
- Why would we?
  Because some are easier to solve than other. E.g., in scheduling exams....
- If A is a total assignment, define $h(A)$ to be a measure of the difficulty of solving problem from A.
- $h(A) = 0$ then A a solution; lower h is better

Variant: Simulated Annealing

- Pick a variable at random and a new value at random.
- If it isn't worse, accept it.
- If it is worse, accept it probabilistically depending on a temperature parameter, T:
  - With current assignment A and proposed assignment A' accept A' with probability $e^{(h(A)-h(A'))/T}$

  Note: $h(A) - h(A')$ is negative if A' is worse.
- Probability of accepting a change:

Temperature	1-worse	2-worse	3-worse
10	0.91	0.81	0.74
1	0.37	0.14	0.05
0.25	0.02	0.0003	0.000006
0.1	0.00005	2×10^{-9}	9×10^{-14}

- Temperature can be reduced.

Random Restart

- A random restart involves reassigning all variables to values at random.
- allows for exploration of a different part of the search space.
- Each run is independent of the others, so probabilities can be derived analytically.
  Suppose each run has a probability of p of finding a solution.
  We do n runs or until a solution is found.
  The probability of n runs failing to find a solution is $(1 - p)^n$
  The probability of finding a solution in n-runs is $1 - (1 - p)^n$

n	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.8$
5	0.410	0.832	0.969	0.9997
10	0.65	0.971	0.9990	0.9999998
20	0.878	0.9992	0.9999991	0.99999999999
50	0.995	0.99999998	0.9999999999999991	1.0

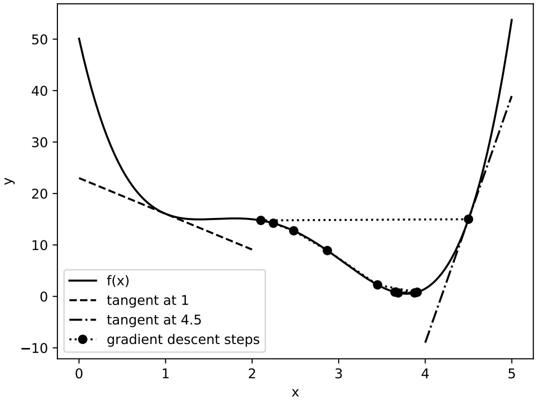
Tabu lists

- To prevent cycling we can maintain a tabu list of the k last assignments.
- Don't allow an assignment that is already on the tabu list.
- If $k = 1$, we don't allow an assignment of to the same value to the variable chosen.
- We can implement it more efficiently than as a list of complete assignments.
- It can be expensive if k is large.

Ordered and Continuous Domains

- When the domains are small or unordered, the neighbors of an assignment can correspond to choosing another value for one of the variables.
- When the domains are large and ordered, the neighbors of an assignment are the adjacent values for one of the variables.
- If the domains are continuous, Gradient descent movies each each variable downhill; proportional to the gradient of the heuristic function in that direction.
  The value of variable X_i goes from v_i to $v_i - \eta \frac{\partial h}{\partial X_i}$.
  η is the step size.
- Neural networks do gradient descent with many parameters (variables) to minimize an error on a dataset. Some large language models have over 10^{12} parameters.

Gradient Descent



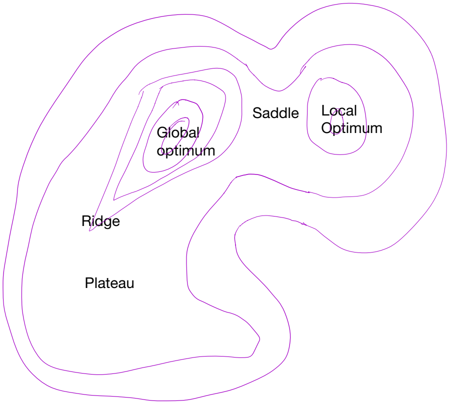
$y = 2 * (x - 1.3) * (x - 1.5) * (x - 2) * (x - 4.5) + 15$

Step size is 0.05 and gradient descent starts at $x = 4.5$.

What if it starts at $x = 5.5$? (Hint: the derivative is more than 4 times larger). What if it starts at $x = 1.5$?

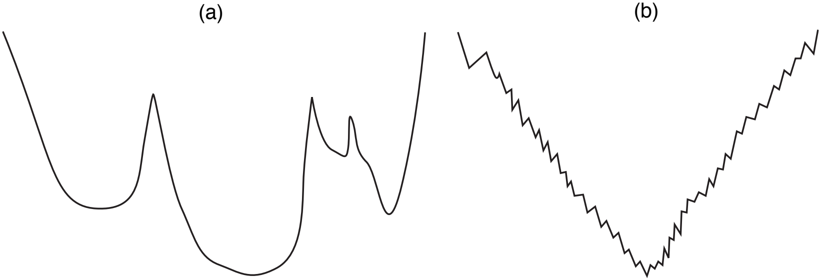
Problems with Greedy Descent

- a local optimum that is not a global optimum
- a plateau where the heuristic values are uninformative
- a ridge is a local minimum where n-step look-ahead might help
- a saddle is a flat area where steps need to change direction



1-Dimensional Ordered Examples

Two 1-dimensional search spaces; small step right or left:



- Which method would most easily find the global minimum?
- What happens in hundreds or thousands of dimensions?
- What if different parts of the search space have different structure?

Parallel Search

A total assignment is called an individual.

- Idea: maintain a population of k individuals instead of one.
- At every stage, update each individual in the population.
- Whenever an individual is a solution, it can be reported.
- Like k restarts, but uses k times the *minimum* number of steps.

Beam Search

- Like parallel search, with k individuals, but choose the k best out of all of the neighbors.
- When $k = 1$, it is greedy descent.
- The value of k lets us limit space and parallelism.
- Problem: lack of diversity of individuals.

Stochastic Beam Search

- Like beam search, but it probabilistically chooses the k individuals at the next generation.
- The probability that a neighbor is chosen is proportional to its heuristic value.
- This maintains diversity amongst the individuals.
- The heuristic value reflects the fitness of the individual.
- Like asexual reproduction: each individual mutates and the fittest ones survive.

Genetic Algorithms

- Like stochastic beam search, but pairs of individuals are combined to create the offspring.
- For each generation:
  - Randomly choose pairs of individuals where the fittest individuals are more likely to be chosen.
  - For each pair, perform a crossover: form two offspring each taking different parts of their parents.
  - Mutate some values.
- Stop when a solution is found.

Crossover

- Given two individuals:

$$X_1 = a_1, X_2 = a_2, \dots, X_m = a_m$$

$$X_1 = b_1, X_2 = b_2, \dots, X_m = b_m$$

- Select i at random.
- Form two offspring:

$$X_1 = a_1, \dots, X_i = a_i, X_{i+1} = b_{i+1}, \dots, X_m = b_m$$

$$X_1 = b_1, \dots, X_i = b_i, X_{i+1} = a_{i+1}, \dots, X_m = a_m$$

- The effectiveness depends on the ordering of the variables.
- Many variations are possible.

Optimization

An optimization problem is given

- a set of variables, each with an associated domain
- an objective function that maps total assignments to real numbers, and
- an optimality criterion, which is typically to find a total assignment that minimizes (or maximizes) the objective function.

Constraint optimization problem

- In a constraint optimization problem the objective function is factored into a sum of soft constraints
- A soft constraint is a function from scope of constraint into non-negative reals (the cost)
- The aim is to find a total assignment that minimizes the sum of the values of the soft constraints.
- Can use systematic search (e.g., A^* or branch-and-bound search)
- Arc consistency can be used to prune dominated values
- Can use local search
- Problem: we can't tell if a value is a global minimum unless we do systematic search