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- Give a factorization, such as

$$P(D) = \sum_C P(D \mid C) \sum_B P(C \mid B) \sum_A P(A)P(B \mid A)$$

it does the innermost sums first, constructing representations of the intermediate factors:

  - $\sum_A P(A)P(B \mid A)$ is a factor on B; call it $f_1(B)$.

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- Lecture covers:
  - Factors and factor arithmetic
  - Variable elimination algorithm

Factors

- A factor is a representation of a function from a tuple of random variables into a number.
- We write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.

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- A factor is a representation of a function from a tuple of random variables into a number.
- We write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.
- You can assign some or all of the variables of a factor:
  - $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in domain(X_1)$, is a factor on $X_2, \ldots, X_j$.
  - $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.

The former is also written as $f(X_1, X_2, \ldots, X_j)_{X_1 = v_1}$, etc.

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

Example factors

$r(X, Y, Z)$:

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Example factors

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f	f	t	0.3
f	f	f	0.7

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t	t	0.1
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$r(X{=}t, Y, Z{=}f)$:

Y	val
t	
f	

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
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Example factors

$r(X, Y, Z)$:

X	Y	Z	val
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f	f	t	0.3
f	f	f	0.7

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t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

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f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$r(X{=}t, Y{=}f, Z{=}f) =$

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
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t	t	0.1
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f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$r(X{=}t, Y{=}f, Z{=}f) = 0.8$

Multiplying factors

The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 * f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 * f_2)(\overline{X}, \overline{Y}, \overline{Z}) = f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	
t	f	t	
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	
f	f	t	
f	f	f	

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B	C	val
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$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
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$f_1 * f_2$:

A	B	C	val
t	t	t	0.03
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t	f	t	0.54
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f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

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t	t	0.3
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f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

Summing out variables

We can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$
\begin{aligned}
& (\sum_{X_1} f)(X_2, \ldots, X_j) \\
= \;\; & f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)
\end{aligned}
$$

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	
f	t	
f	f	

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	
f	f	

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Clicker Question

If $f(W, X, Y, Z)$ is a factor on variables $\{W, X, Y, Z\}$, then $f(W, X = 3, Y = \mathit{true}, Z)$ is a factor on

A $\{W, X, Y, Z\}$

B $\{X, Y\}$

C $\{W, Z\}$

D $\{\}$

E none of the above

Clicker Question

If $f(W, X, Y, Z)$ is a factor on variables $\{W, X, Y, Z\}$, then $f(W = 17, X = 3, Y = \mathit{true}, Z = \mathit{false})$ is a factor on

A $\{W, X, Y, Z\}$

B $\{X, Y\}$

C $\{W, Z\}$

D $\{\}$

E none of the above

Clicker Question

If f is a factor on $\{W, X, Y\}$ and
g is a factor on $\{W, U\}$
$(f * g)$ is a factor on

A $\{W, X, Y, U\}$

B $\{X, Y, U\}$

C $\{W\}$

D $\{f, g, W, X, Y, U\}$

E there is not enough information to tell

Clicker Question

If $f(W{=}3, X{=}4, Y{=}5) = 10$ and
$g(W{=}3, U{=}12) = 15$
$(f * g)(W{=}3, X{=}4, Y{=}5, U{=}12) =$

A a factor on $\{W, X, Y, U\}$

B 25

C 150

D none of the above

E there is not enough information to tell

Exercise

Given factors:

s:

A	val
t	0.75
f	0.25

t:

A	B	val
t	t	0.6
t	f	0.4
f	t	0.2
f	f	0.8

o:

A	val
t	0.3
f	0.1

What are the following a function of?

i) $s * t$

A $\{A\}$

B $\{B\}$

C $\{A, B\}$

D $\{\}$

E none of the above

Exercise

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A	val
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What are the following a function of?

i) $s * t$

ii) $\sum_B (s * t)$

A $\{A\}$

B $\{B\}$

C $\{A, B\}$

D $\{\}$

E none of the above

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What are the following a function of?

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iii) $s * o$

A $\{A\}$

B $\{B\}$

C $\{A, B\}$

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What are the following a function of?

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iii) $s * o$

iv) $\sum_A s * t * o$

A $\{A\}$

B $\{B\}$

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D $\{\}$

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t:

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f	f	0.8

o:

A	val
t	0.3
f	0.1

What are the following a function of?

i) $s * t$

ii) $\sum_B (s * t)$

iii) $s * o$

iv) $\sum_A s * t * o$

v) $\sum_B (\sum_A s * t * o)$

A $\{A\}$

B $\{B\}$

C $\{A, B\}$

D $\{\}$

E none of the above

Queries and Evidence

- To compute the posterior probability of query Q given evidence $E = e$:

$$P(Q \mid E = e)$$

$$=$$

Queries and Evidence

- To compute the posterior probability of query Q given evidence $E=e$:

$$\begin{aligned} & P(Q \mid E=e) \\ = \ & \frac{P(Q, E=e)}{P(E=e)} \\ = \ & \end{aligned}$$

Queries and Evidence

- To compute the posterior probability of query Q given evidence $E = e$:

$$
\begin{aligned}
P(Q \mid E = e) \\
&= \frac{P(Q, E = e)}{P(E = e)} \\
&= \frac{P(Q, E = e)}{\sum_Q P(Q, E = e)}.
\end{aligned}
$$

Queries and Evidence

- To compute the posterior probability of query Q given evidence $E = e$:

$$
\begin{aligned}
& P(Q \mid E = e) \\
& \quad = \frac{P(Q, E = e)}{P(E = e)} \\
& \quad = \frac{P(Q, E = e)}{\sum_Q P(Q, E = e)}.
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- So the computation reduces to the probability of $P(Q, E = e)$

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&\quad = \frac{P(Q, E = e)}{\sum_Q P(Q, E = e)}.
\end{aligned}
$$

- So the computation reduces to the probability of $P(Q, E = e)$
- then normalize at the end.

Probability of a conjunction

- The variables of the belief network are $X_1, \ldots, X_n$.
- The evidence is $Y_1 = v_1, \ldots, Y_j = v_j$
- To compute $P(Q, Y_1 = v_1, \ldots, Y_j = v_j)$:

Probability of a conjunction

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  we add the other variables,
  $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} - \{Q\} - \{Y_1, \ldots, Y_j\}$.
  and sum them out.

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  and sum them out.
- We order the Z_i into an elimination ordering.

$$P(Q, Y_1 = v_1, \ldots, Y_j = v_j)$$

$$=$$

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  and sum them out.
- We order the Z_i into an elimination ordering.

$$P(Q, Y_1 = v_1, \dots, Y_j = v_j)$$
$$= \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \dots, X_n)_{Y_1 = v_1, \dots, Y_j = v_j}.$$
$$=$$

Probability of a conjunction

- The variables of the belief network are $X_1, \ldots, X_n$.
- The evidence is $Y_1 = v_1, \ldots, Y_j = v_j$
- To compute $P(Q, Y_1 = v_1, \ldots, Y_j = v_j)$:
  we add the other variables,
  $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} - \{Q\} - \{Y_1, \ldots, Y_j\}$.
  and sum them out.
- We order the Z_i into an elimination ordering.

$$
\begin{aligned}
& P(Q, Y_1 = v_1, \ldots, Y_j = v_j) \\
= & \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
= & \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i \mid parents(X_i))_{Y_1 = v_1, \ldots, Y_j = v_j}.
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- Distribute out those factors that don't involve Z_1.

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- Multiply the remaining factors.
- Normalize by dividing the resulting factor $f(Q)$ by $\sum_Q f(Q)$.

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To sum out a variable Z_j from a product $f_1, \ldots, f_k$ of factors:

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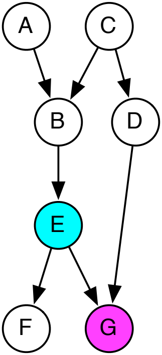
Then:

$$\sum_{Z_j} f_1 * \cdots * f_k = f_1 * \cdots * f_i * \left(\sum_{Z_j} f_{i+1} * \cdots * f_k \right).$$

- Explicitly construct a representation of the rightmost factor. Replace the factors $f_{i+1}, \dots, f_k$ by the new factor.

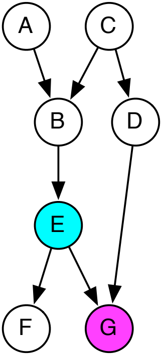
Example

$$P(E \mid g) \quad =$$



Example

$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$



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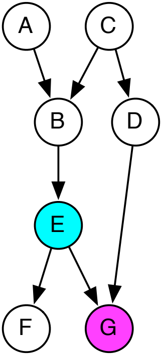

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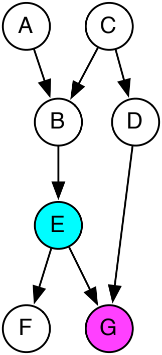

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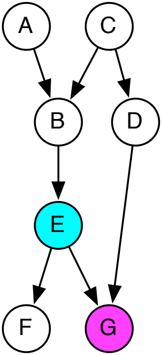
Example



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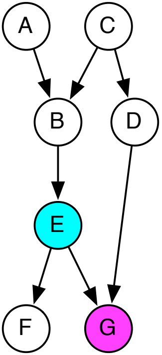

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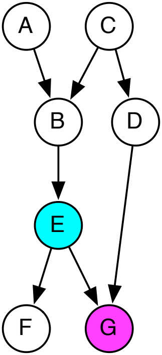

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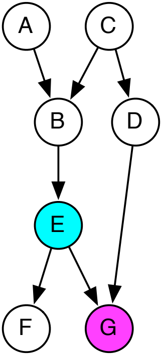

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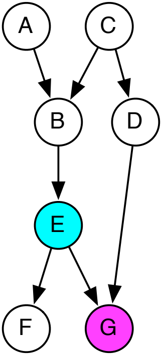

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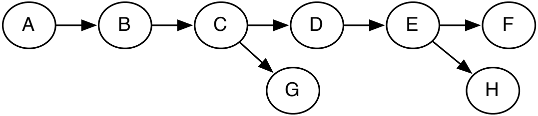
Variable Elimination example



Query: $P(G \mid f)$; elimination ordering: A, H, E, D, B, C

$$P(G \mid f) \quad \propto$$

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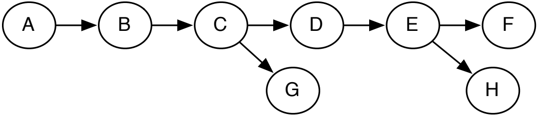

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$$P(G \mid f) \propto \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B \mid A)P(C \mid B) P(D \mid C)P(E \mid D)P(f \mid E)P(G \mid C)P(H \mid E)$$

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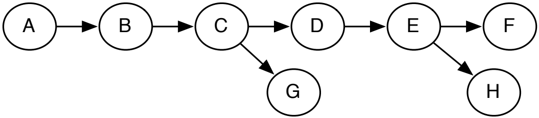

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$$= \sum_C \left(\sum_B \left(\sum_A P(A)P(B \mid A) \right) P(C \mid B) \right) P(G \mid C) \left(\sum_D P(D \mid C) \left(\sum_E P(E \mid D)P(f \mid E) \sum_H P(H \mid E) \right) \right)$$

Clicker Question

In variable elimination with factors:

$$f_0(W), f_1(W, X), f_2(X, Y), f_3(Y, Z)$$

If variable X is eliminated (summed out) first which factors are multiplied when summing X out:

A none of them

B f_1 and f_2

C f_0, f_1 and f_2

D f_1, f_2 and f_3

E all of them

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B f_3 and $\sum_X f_0 * f_1 * f_2$

C f_0, f_1, f_2, f_3 and $\sum_X f_1 * f_2$

D f_0, f_3 and $\sum_X f_1 * f_2$

E all of f_0 ,f_1, f_2, f_3

Pruning Irrelevant Variables (Belief networks)

Suppose you want to compute $P(X \mid e_1 \dots e_k)$:

- Prune any variables that have no observed or queried descendents.
- Connect the parents of any observed variable.
- Remove arc directions.
- Remove observed variables.
- Remove any variables not connected to X in the resulting (undirected) graph.

Clicker Question

Given evidence and a query, which variables can be pruned before running variable elimination

A all of the variables that are not observed or queried

B those variables after the query variable in the total ordering of variables that defines the belief network

C all variables that are not observed or queried or are parents of queried variables

D all variables that are not observed or queried and have no observed or queried descendents

E none of the variables

Variable Elimination and Recursive Conditioning

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- Variable elimination is the dynamic programming variant of recursive conditioning.
- Recursive Conditioning is the search variant of variable elimination.
- They do the same additions and multiplications.
- Space and time complexity $O(nd^t)$, for n variables, domain size d, and treewidth t.
  – treewidth is the number of variables in the smallest factor.
  It is a property of the graph and the elimination ordering.
- Recursive conditioning never modifies or creates factors; it only evaluates them.